MATH 230 Homework #2 Solutions

1. $$\lim_{N\to\infty} \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \lim_{N\to\infty} \frac{k!}{x!(k-x)!} \frac{(N-k)!}{(n-x)![N-k-(n-x)]!} \frac{n!(N-n)!}{N!}$$

$$= \lim_{N\to\infty} \frac{n!}{x!(n-x)!} \frac{N^n}{N^x N^{n-x}} \frac{k!}{(k-x)!} \frac{(N-k)!}{[N-k-(n-x)]!} \frac{(N-n)!}{N!}$$

$$= \lim_{N\to\infty} \binom{n}{x} \frac{k(k-1)\cdots(k-x+1)}{N^x} \cdot \frac{(N-k)(N-k-1)\cdots(N-k-(n-x)+1)}{N^{n-x}} \frac{N^n}{N(N-1)\cdots(N-n+1)}$$

$$= \lim_{N\to\infty} \binom{n}{x}\left(\frac{k}{N}\right)\left(\frac{k}{N}-\frac{1}{N}\right)\cdots\left(\frac{k}{N}-\frac{k-x}{N}\right)\left(\frac{N-k}{N}\right)\left(\frac{N-k}{N}-\frac{1}{N}\right)\cdots\left(\frac{N-k}{N}-\frac{n+x-1}{N}\right)$$

$$\cdot\left(\frac{N}{N}\right)\left(\frac{N}{N-1}\right)\cdots\left(\frac{N}{N-n+1}\right)$$

$$= \binom{n}{x}\left(\frac{k}{N}\right)^x\left(1-\frac{k}{N}\right)^{n-x} = \binom{n}{x}p^x(1-p)^{n-x}$$

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2. $P(X=k) = \frac{e^{-\mu}\mu^k}{k!}$ $\quad \ln P(X=k) = -\mu + k\ln\mu - \ln(k!)$

$$\frac{d\ln P(X=k)}{d\mu} = -1+\frac{k}{\mu} = 0 \Rightarrow k=\mu$$

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3. a) $P(Y>0.1) = \int_{0.1}^{1} dy = 1-0.1 = 0.9$

b) $P(Y>0.2 \mid Y>0.1) = \frac{P(Y>0.2)}{P(Y>0.1)} = \frac{0.8}{0.9} = \frac{8}{9}$

c) $P(Y>0.3 \mid Y>0.2, Y>0.1) = \frac{P(Y>0.3)}{P(Y>0.2)} = \frac{0.7}{0.8} = \frac{7}{8}$

d) $P(Y>0.3, Y>0.2, Y>0.1) = P(Y>0.3) = 0.7$

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4. We'll find v such that

$\int_v^1 5(1-x)^4 dx = 0.01 \Rightarrow (1-v)^5 = 0.01 \Rightarrow v = 1-\sqrt[5]{0.01} = 0.602$ thousands of gallons.

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5. $X \sim N(\mu,\sigma^2)$

$F_Y(y) = P(Y\le y) = P(e^X \le y) = P(X\le \ln y) = F_X(\ln y)$

$f_Y(y) = F_Y'(y) = \frac{1}{y} f_X(\ln y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left\{\frac{-(\ln y-\mu)^2}{2\sigma^2}\right\} \quad y>0$

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6. X: time to run a mile distance $\quad X\sim N(\mu=450, \sigma=40)$

$P(X\le x_0) = 0.10$, $P\left(Z\le \frac{x_0-450}{40}\right) = 0.10$, $P(Z<z_0) = 0.10$

$\Rightarrow z_0 = -1.28 \quad \frac{x_0-450}{40} = -1.28 \Rightarrow x_0 = 450-(40)(1.28) = 398.8$

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7. $T \sim exp(5) \quad P(T>8) = e^{-\frac{8}{5}} = 0{,}2$

Y: # of components survived more than 8 years in 5. $Y \sim Bin(n=5, p=0{,}2)$

$P(Y \geq 2) = 1 - P(Y \leq 1) = 1 - \binom{5}{0}(0{,}2)^0(0{,}8)^5 - \binom{5}{1}(0{,}2)(0{,}8)^4 = 1 - (0{,}8)^5 - (0{,}8)^4$

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8. X: # of 6's in 1000 rolls. $X \sim Bin(n=1000, p=\frac{1}{6})$

Using normal app. to the Binomial dist.;

$$P(150 \leq X \leq 200) \sim P\left(\frac{150 - 0{,}5 - \frac{1000}{6}}{\sqrt{\frac{5000}{36}}} \leq Z \leq \frac{200 + 0{,}5 - \frac{1000}{6}}{\sqrt{\frac{5000}{36}}}\right)$$

$$= P(-1{,}46 \leq Z \leq 2{,}87)$$

$$= F(2{,}87) - F(-1{,}46) = 0{,}9979 - 0{,}0721 = 0{,}9258$$

If 6 appears exactly 200 times, our interest # of 5's in other 800 rolls.

$Y \sim Bin(n=800, p=\frac{1}{5})$, then

$$P(Y<150) = P(Y \leq 149) \sim P\left(Z \leq \frac{149{,}5 - \frac{800}{5}}{\sqrt{\frac{(800)(4)}{25}}}\right) = P(Z < -0{,}93) = 0{,}1762$$

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9. $P(X<100 \mid X>90) = 0{,}15$

$$P(X<100 \mid X>90) = \frac{P(90<X<100)}{P(X>90)} = \frac{\int_{90}^{100} 2\alpha x e^{-\alpha x^2} dx}{\int_{90}^{\infty} 2\alpha x e^{-\alpha x^2} dx} = \frac{e^{-\alpha(90)^2} - e^{-\alpha(100)^2}}{e^{-\alpha(90)^2}} = 0.15$$

$$\Rightarrow e^{-\alpha(90)^2}(0{,}15) = e^{-\alpha(90)^2} - e^{-\alpha(100)^2} \Rightarrow (1-0{,}15)e^{-\alpha(90)^2} = e^{-\alpha(100)^2}$$

$$\Rightarrow \ln(0{,}85) - \alpha(90)^2 = -\alpha(100)^2 \Rightarrow \ln(0{,}85) = (-10000 + 8100)\alpha \Rightarrow \alpha = \frac{\ln(0{,}85)}{1900}$$

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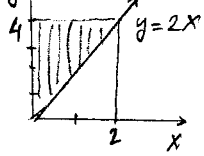
10. X: CPU time $\sim$ Gamma $(\alpha=3, \beta=2)$ $\quad f(x) = \frac{1}{\Gamma(3) 2^3} x^{3-1} e^{-\frac{x}{2}} = \frac{1}{16} x^2 e^{-\frac{x}{2}} \quad x>0$

$P(X>1) = \int_1^{\infty} \frac{x^2}{16} e^{-\frac{x}{2}} dx = \frac{1}{8}\left(e^{-\frac{1}{2}} + 4e^{-\frac{1}{2}} + 8e^{-\frac{1}{2}}\right)$ (by integration by parts)

11.



a) $\int_0^2 \int_{2x}^4 cx\,dy\,dx = c(2x^2 - \frac{2}{3}x^3)\Big|_0^2 = c\,\frac{8}{3} = 1 \Rightarrow c = \frac{3}{8}$

b) $f_X(x) = \int_{2x}^4 \frac{3}{8} x\,dy = \frac{3}{2}x - \frac{3}{4}x^2 \quad 0<x<2, \quad f_Y(y) = \int_0^{\frac{y}{2}} \frac{3}{8} x\,dx = \frac{3y^2}{64} \quad 0<y<4$

c) $E(X) = \int_0^2 x(\frac{3}{2}x - \frac{3}{4}x^2)dx = 1, \quad E(Y) = \int_0^4 \frac{3}{64} y^3 dy = 3, \quad E(XY) = \int_0^2 \int_{2x}^4 xy \frac{3}{8} x\,dy\,dx = \frac{16}{5}$

$Cov(X,Y) = \frac{16}{5} - (1)(3) = \frac{16-15}{5} = \frac{1}{5}$

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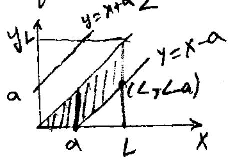
12. X: location of accident, Y: location of ambulance at the moment of accident.

$f(x) = \frac{1}{L} \quad 0<x<L, \quad f(y) = \frac{1}{L} \quad 0<y<L, \quad f(x,y) = \frac{1}{L^2} \quad 0<x<L, 0<y<L$



$$F_{|Y-X|}(a) = P(|Y-X| \leq a) = P(-a \leq Y - X \leq a) = P(X-a \leq Y \leq X+a)$$

$$= 2P(X-a \leq Y \leq X) = \frac{2}{L^2}\left[\int_0^a \int_0^x dy\,dx + \int_a^L \int_{x-a}^x dy\,dx\right] = \frac{2}{L^2}\left(\frac{a^2}{2} + aL - a^2\right)$$

$$= \frac{2a}{L} - \frac{a^2}{L^2} \quad a>0$$
