NAME: ....

Date: October 19, 2007
Time: 18:00-19:40
Instructor: Dilek Güvenç

MATH 230 MIDTERM EXAM I

IMPORTANT

1 Check that there are 5 questions in your booklet.
2 Do **NOT** use your mobile phone as a calculator. Turn it off during the exam.
3 Show all your work. Correct results without sufficient explanation and correct notation might not get full credit.
4 Write your name on each page.

1	2	3	4	5	TOTAL
20	20	20	20	20	100

GOOD LUCK!

1. Suppose that a random variable X takes values of 0, 1 and 2. For some constant c, P(X= i) = cP(X= i-1) i=1,2 . Find the expected value of X in terms of c.

$$P(X=1) = c\,P(X=0)$$

$$P(X=2) = c\,P(X=1) = c\,[c\,P(X=0)] = c^2 P(X=0)$$

$$P(X=0) + P(X=1) + P(X=2) = 1$$

$$P(X=0) + c\,P(X=0) + c^2 P(X=0) = 1$$

$$\Rightarrow \quad P(X=0) = \frac{1}{1+c+c^2}$$

x	0	1	2
$f(x)$	$\frac{1}{1+c+c^2}$	$\frac{c}{1+c+c^2}$	$\frac{c^2}{1+c+c^2}$

$$E(X) = \sum_{x=0}^{2} x f(x) = \frac{1}{1+c+c^2}\left[0 + c + 2c^2\right] = \frac{c+2c^2}{1+c+c^2}$$

RESERVE 3

NAME:..........................

2. Consider a time division multiple access (TDMA) wireless system, where the base transceiver system of each cell has n repeaters. Each base repeater provides m channels, thus there are mn channels in the system. A base repeater subject to failure. In order to evaluate the impact of such a failure on the performability of the system one should know the number of ungoing talking channels on the failed base repeater. Suppose channels are allocated to the users randomly. If there are k talking channels in the whole system, find the probability that i ($i \leq \min(m,k)$) talking channels reside in the failed base repeater.

$$n(S) = \binom{mn}{k}$$

$A = \{ i$ talking channels are in failled base repeater$\}$

$$n(A) = \binom{m}{i}\binom{mn-m}{k-i} = \binom{m}{i}\binom{m(n-1)}{k-i}$$

$$P(A) = \frac{\binom{m}{i}\binom{m(n-1)}{k-i}}{\binom{mn}{k}} \qquad i = 1, 2 \ldots \min(m,k)$$

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RESERVE

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3. a) If A and B are independent events, show that A' and B' are independent events. (A' and B' are complement of A and B, respectively)
(10 points)

b) Suppose a balanced coin is tossed two times. Define the following events:
A ={ Head appears on the first toss}
B ={ Head appears on the second toss}
C ={ Both tosses yield the same outcome}
Are A, B and C independent?
(10 points)

a) A and B are independent $\Rightarrow P(A \cap B) = P(A)P(B)$

We'll show that $P(A' \cap B') = P(A') . P(B')$

$$
\begin{aligned}
P(A' \cap B') = P(A \cup B)' &= 1 - P(A \cup B) \\
&= 1 - P(A) - P(B) + P(A \cap B) \\
&= P(A') - P(B) + P(A)P(B) \\
&= P(A') - P(B)[1 - P(A)] \\
&= P(A') - P(B)P(A') \\
&= P(A')[1 - P(B)] \\
&= P(A') . P(B')
\end{aligned}
$$

b) $S = \{HH, HT, TH, TT\}$

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4} = P(A) . P(B) = \frac{1}{2} \cdot \frac{1}{2}$

$P(A \cap C) = \frac{1}{4} = P(A) . P(C) = \frac{1}{2} \cdot \frac{1}{2}$.

$P(B \cap C) = \frac{1}{4} = P(B) . P(C) = \frac{1}{2} \cdot \frac{1}{2}$.

A, B and C are pairwise independent, but:

$P(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)P(C)$

$\Rightarrow$ A, B and C are not independent.

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RESERVE

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4. Suppose that when in flight, airplane engines operate with probability p independently from engine to engine. An airplane will be able to make a successful flight if at least 50 percent of its engines operate. For what values of p four-engine plane preferable to a two-engine plane?

With 4-engine plane

$P(\text{successful flight}) = \binom{4}{2}p^2(1-p)^2 + \binom{4}{3}p^3(1-p) + \binom{4}{4}p^4$

with 2-engine plane

$P(\text{successful flight}) = \binom{2}{1}p(1-p) + \binom{2}{2}p^2$

$6p^2(1-p)^2 + 4p^3(1-p) + p^4 > 2p(1-p) + p^2 \qquad p \neq 0$

$6p(1-2p+p^2) + 4p^2 - 4p^3 + p^3 > 2 - 2p + p$

$6p - 12p^2 + 6p^3 + 4p^2 - 3p^3 - 2 + p > 0$

$3p^3 - 8p^2 + 7p - 2 > 0$

$(p-1)(3p^2 - 5p + 2) > 0$

$(p-1)(3p-2)(p-1) = (p-1)^2(3p-2) > 0$

Since $(p-1)^2 > 0 \Rightarrow 3p - 2 > 0 \Rightarrow p > \frac{2}{3}$

RESERVE 6

NAME: ....

5. Assume that the probability of error-free transmission of a message over a communication channel is 0.8. If a message is not transmitted correctly, a retransmission is initiated. This procedure is repeated until a correct transmission occurs. Such a channel is often called a "feedback channel". Assuming that successive transmissions are independent,

a) what is the probability that no retransmissions are required? **(6 points)**
b) what is the probability that exactly two retransmissions are required? **(6 points)**
c) what is the probability that third error-free transmission occurs before the fifth transmission? **(8 points)**

(Define the random variable, determine its probability distribution and solve.)

X: # of transmissions needed to have first correct transmission.

$X \sim$ Geometric $(p = 0.8)$

a) $P(X=1) = 0.8$

b) $P(X=3) = (0.2)^2(0.8) = 0.032$

c) Y: # of transmissions required to have the 3-rd error-free transmission.

$Y \sim$ Neg. Bin $(r=3,\ p=0.8)$

$$P(Y<5) = \sum_{y=3}^{4} \binom{y-1}{3-1} (0.8)^3 (0.2)^{y-3}$$

$$= (0.8)^3 + 3(0.8)^3(0.2)$$

$$= (0.8)^3(1.6)$$

$$= 0.8192$$